The Provable Security Methodology

Pooya Farshim

Queen's University Belfast/École Normale Supérieure

Mathematical Aspects of Computer Science – Foundations of Cryptography

متدولوژیِ امنیت قابل اثبات

پویا فرشیم

دانشگاه کوئینز بلفاست – اکول نرمال سوپریور

جنبه‌های ریاضی علوم کامپیوتری – مبانی رمز

Unreadable Effectiveness of Mathematics

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2 R.W. Hamming. The unreasonable effectiveness of mathematics. *The American Mathematical Monthly*, 1980.

Cryptography, *n*

The Oxford English Dictionary defines:

> **cryptography**, *n.* **1.** The art or practice of writing in code or cipher;

Ad Hoc Protocol Deign

Problem

Proposed Solution

Bug!

Revised Solution

...

Implementation

Bug!

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Source: Bellare & Rogaway. Introduction to Modern Cryptography. 2005.

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The entry continues

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A broader definition that's perhaps better suited for us:

> **cryptology**: the scientific/mathematical study of systems in the presence of adversarial behavior.

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(2) Formulation of a precise definition of security
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(4) Proof that an instance of (1) is secure wrt. definition (2) relative to assumptions (3).

See "Post-Modern Cryptography" [Gol06] for a (vehement!) defense.

The Role of Definitions

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1. $\underline{\mathsf{Gen}(1^\lambda)}$: is randomized and outputs (fk, td).
2. $\underline{\mathsf{F}(fk, x)}$: is deterministic; given fk and $x \in \mathsf{D}_{fk}$ outputs $y \in \mathsf{R}_{fk}$.
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Or demand $\mathsf{F}^{-1}(td, \cdot)$ **samples** a random pre-image.

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A cryptosystem is called secure if

Any adversary of specified resources has small advantage (according to the metric) in achieving the specified break.

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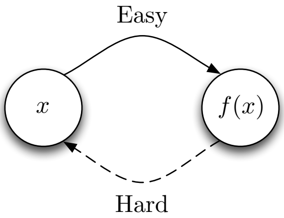

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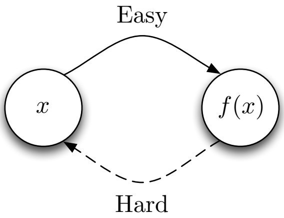

Formalization involves multiple choices:

- Computing **a** pre-image vs. **the** pre-image: Is $x \mapsto 0$ useful?
- How is $f(x)$ sampled? Is it sampled at all?
- Some pre-image can always be found in exponential time.
- And can always guess x with nonzero probability.
- How "small" a probability do we want?

Negligible Functions

A negligible function $\mu : \mathbb{N} \longrightarrow \mathbb{R}$ approaches zero faster that $1/\mathrm{poly}(\lambda)$ for any $\mathrm{poly}(\lambda)$.[1] In other terms

$$\mu(\lambda) \in \lambda^{-\omega(1)} \ .$$

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Define

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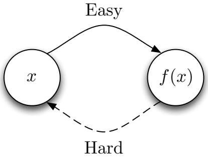

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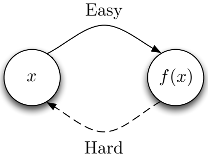

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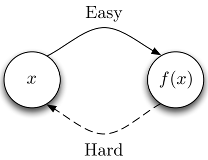

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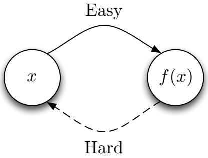

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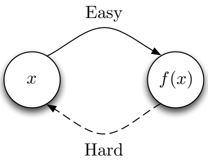

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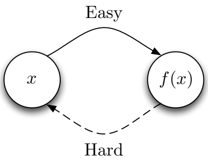

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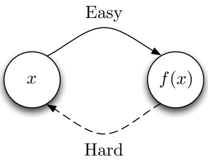

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Require:

$$\forall \text{ ppt TMs } A \ : \ \mathbf{Adv}^{\mathrm{ow}}_{\mathsf{FF},A}(\lambda) \in \mathrm{Negl}\ .$$

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As we shall see: “OWFs exist” is a very safe assumption.

The "Right" Notion?

- Weak OWFs: There is a $\mathrm{poly}(\lambda)$ such that for all ppt A:

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Can we relate these notions?

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- Ultimately: How accepting the community is, and how fruitful a definition turns out to be.

(3) Assumptions [GK15]

Definition (Crypto Assumption)

A (search) cryptographic assumption is an interactive protocol

$$\langle \mathsf{C}(1^\lambda), A(1^\lambda)\rangle$$

between a challenger C and an adversary A that terminates with C outputting T/F. The assumption holds in the average case if for all ppt A

$$\mathbf{Adv}_{\mathsf{C},A}(\lambda) := \Pr[\langle \mathsf{C}(1^\lambda), A(1^\lambda)\rangle] \in \mathrm{Negl}\ .$$

It holds in the worst case if for all non-uniform pt A

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When C is ppt, we call the assumption **falsifiable** [Nao03].

Simple(r) Assumptions

Definition (Search complexity assumption)

Protocol $\langle \mathsf{C}, A \rangle$ for some ppt (D, R) can be written as:

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To what extent are these existentially equivalent?

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$$\mathbf{Adv}_{\mathsf{C},A}(\lambda) \notin \mathrm{Negl} \implies \mathbf{Adv}_{\mathsf{C},B}(\lambda) = 1 \ .$$

Example: Discrete Logs: Invert $(g, x) \mapsto (g, g^x)$.

- Randomize (g, g^x) as (g^r, g^{xs}) for random (r, s).
- Given y, return $\frac{ry}{s}$.

For OWFs: Related to basing OWFs on $\mathbf{NP}$ hardness.

Average-Case and Worst-Case

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What about other primitives?

Can we base average-case hash functions on worst-case hash functions?

Universality of OWF

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Formalize universality. Which primitives are universal?

(4) Security Results

Results in provable security often look like this:

Theorem

If scheme D is secure wrt. its definition of security, then construction C is secure wrt. its definition of security. More precisely, for any $t(\lambda)$-time A, there is a $t'(\lambda)$-time adversary B such that:

$$\mathbf{Adv}^{\mathrm{sec1}}_{C,A}(\lambda) \leq \tau(\lambda) \cdot \mathbf{Adv}^{\mathrm{sec2}}_{D,B}(\lambda) + \mu(\lambda)$$

for a (polynomial) function $\tau(\lambda)$ and a negligible function $\mu(\lambda)$.

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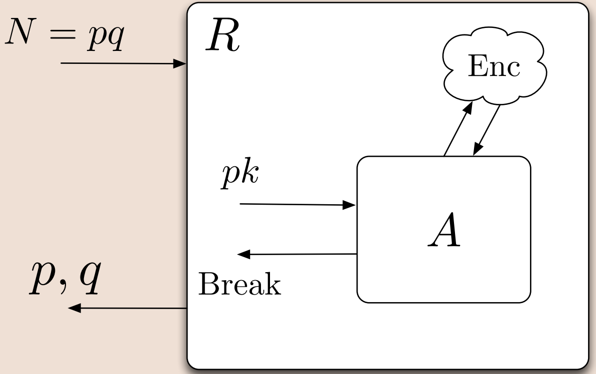
Guides us how to set key sizes.

Proofs via Reductions

Proof.



□

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Caveats:

- Provable security never proves security in an absolute sense.
- Security is always relative to the assumptions and the model.
- Definitions, assumptions, protocols and proofs change!

Collision Resistance

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Require:

$$\forall \text{ ppt } A \ : \ \mathbf{Adv}^{\mathrm{cr}}_{\mathsf{FF},A}(\lambda) := \Pr[\mathrm{CR}^{A}_{\mathsf{FF}}(1^\lambda)] \in \mathrm{Negl} \ .$$

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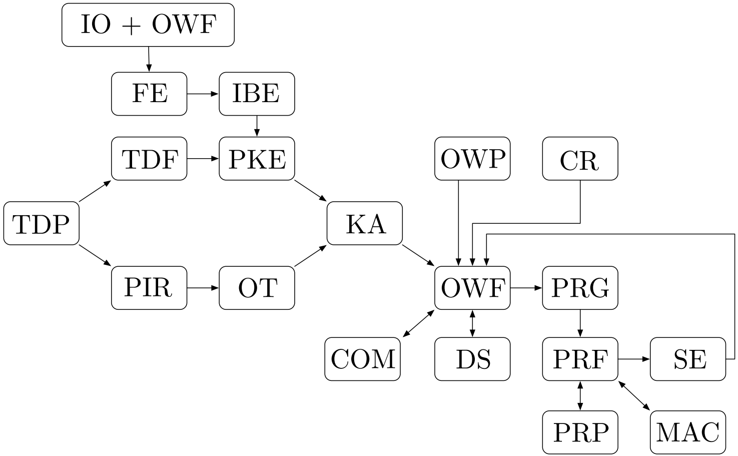
OWF is (strictly) weaker than CRF as an assumption.

Landscape of Primitives



Thanks!