A (Second) Introduction to Provable Security

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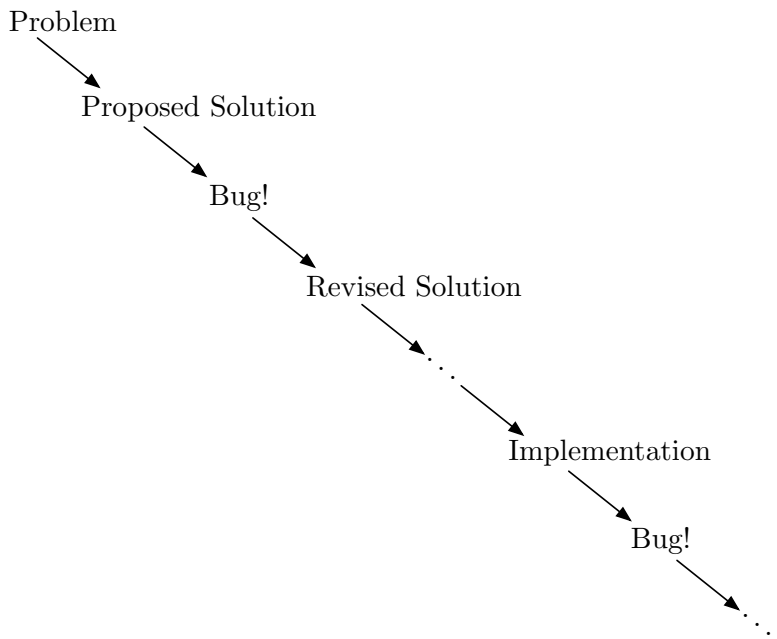
Defining Cryptography

The Oxford English Dictionary defines:

> **cryptography**, *n.* **1.** The art or practice of writing in code or cipher;



Source: Bellare & Rogaway. Introduction to Modern Cryptography. 2005.

Defining Cryptography

The entry continues

> the science of encryption; the branch of cryptology concerned with this (cf. cryptanalysis *n.*). More generally: the study of codes and ciphers; cryptology.

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A broader definition that's perhaps better suited for us:

> **cryptology**: the scientific/mathematical study of systems in the presence of adversarial behavior.

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The Definition-Theorem-Proof approach in Cryptography.

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(3) Statement of **Assumptions** (as instances of (1) and (2)).

(4) A **Proof** that an instance/construction of (1) is secure wrt. definition (2) relative to assumptions (3).

(Roughly reflects the typical structure of a crypto paper.)

Syntax and Correctness

(1) Syntax and Correctness.

Symmetric Encryption

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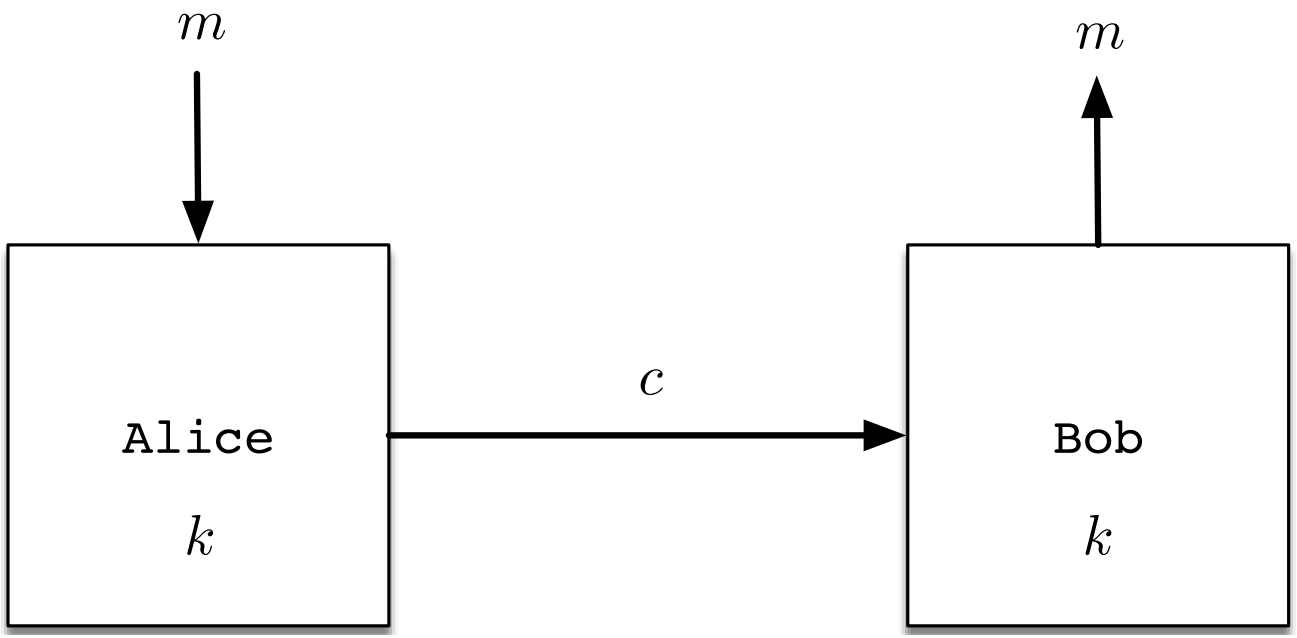

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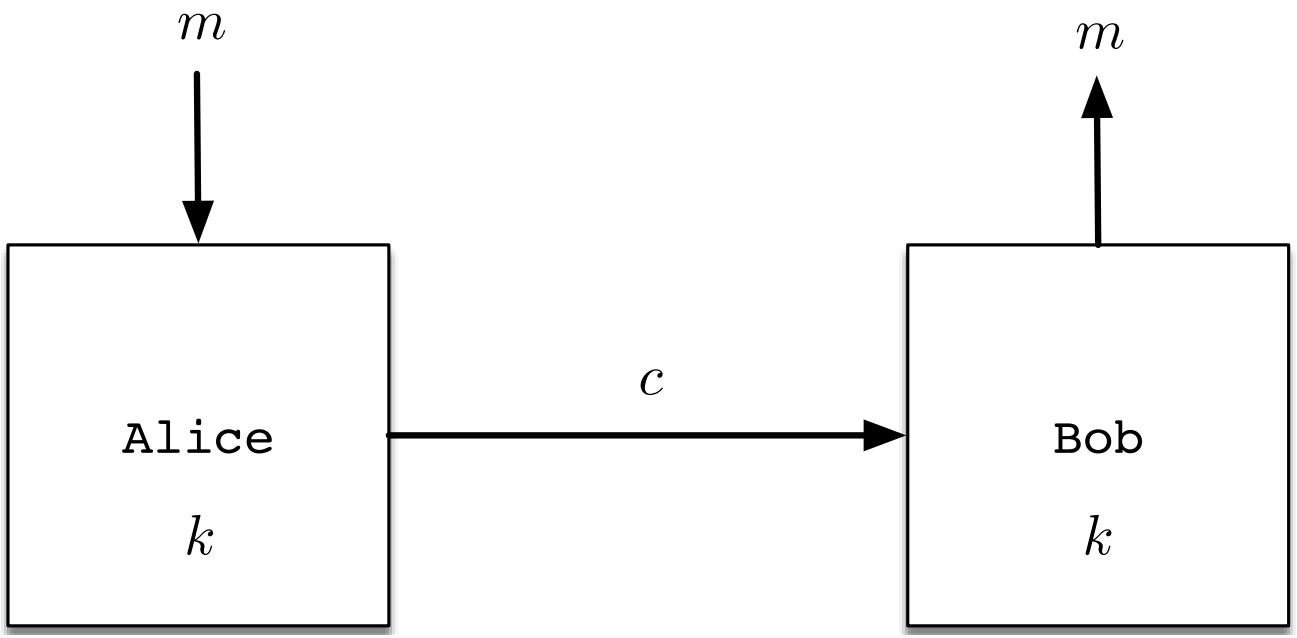

Formalize available I/O via three algorithms

$$\mathsf{SE} := (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$$

and say that Dec undoes Enc.

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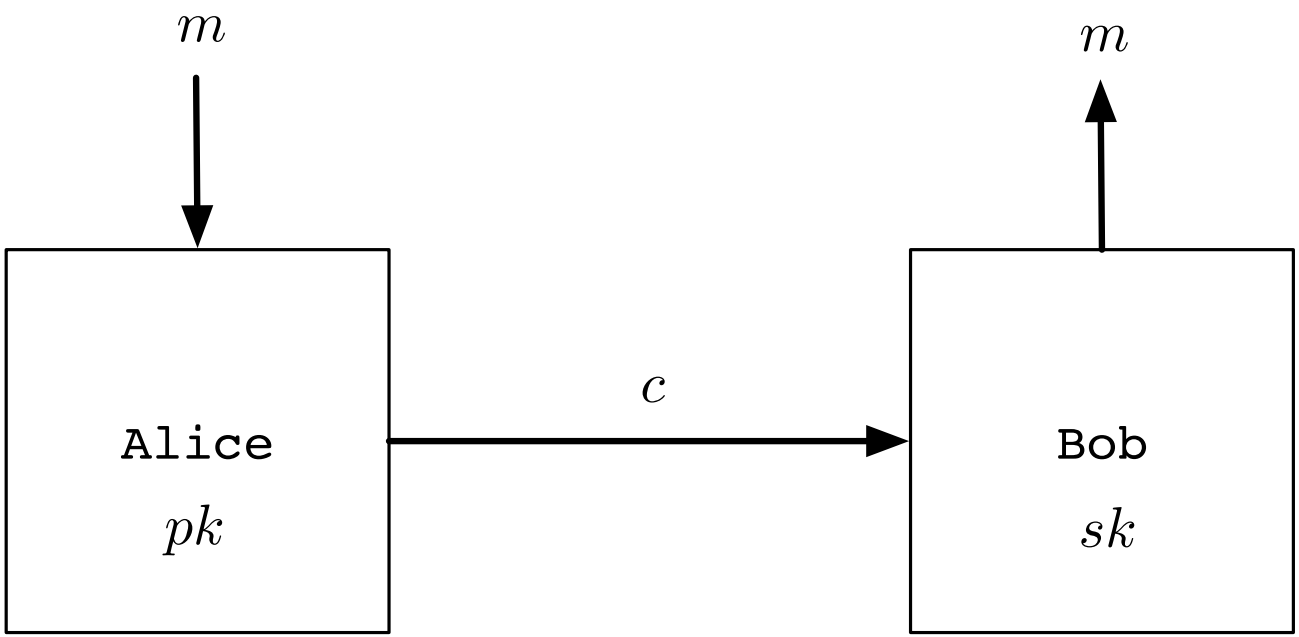
Correctness: For any $\lambda \in \mathbb{N}$, any $k \leftarrow_{\$} \mathsf{Gen}(\lambda)$, and any $m \in \{0,1\}^*$

$$\mathsf{Dec}(k, \mathsf{Enc}(k, m)) = m \ .$$

Public-Key Encryption



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Syntax

- We cannot talk about what an object is supposed to do before defining it.
- Syntax and correctness clarify what the problem at hand is.
  - Available interfaces
  - The input/output behavior
  - The Expected functionality
- It sets the ground for security definitions and proofs.

(2) Security Definitions.

When Is a Symmetric Encryption Secure?

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Clearly correct.

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Clearly correct. Also secure:

It's indeed impossible to find k from c (or even many c's).

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Also secure:

Impossible to find m_2 from (a single!) (c_1, c_2).

But the left half of the message leaks in the clear.

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This scheme is secure:

Impossible to find m from c as c is random.

But given $m \oplus k$ and $m' \oplus k$, one can learn $m \oplus m'$.

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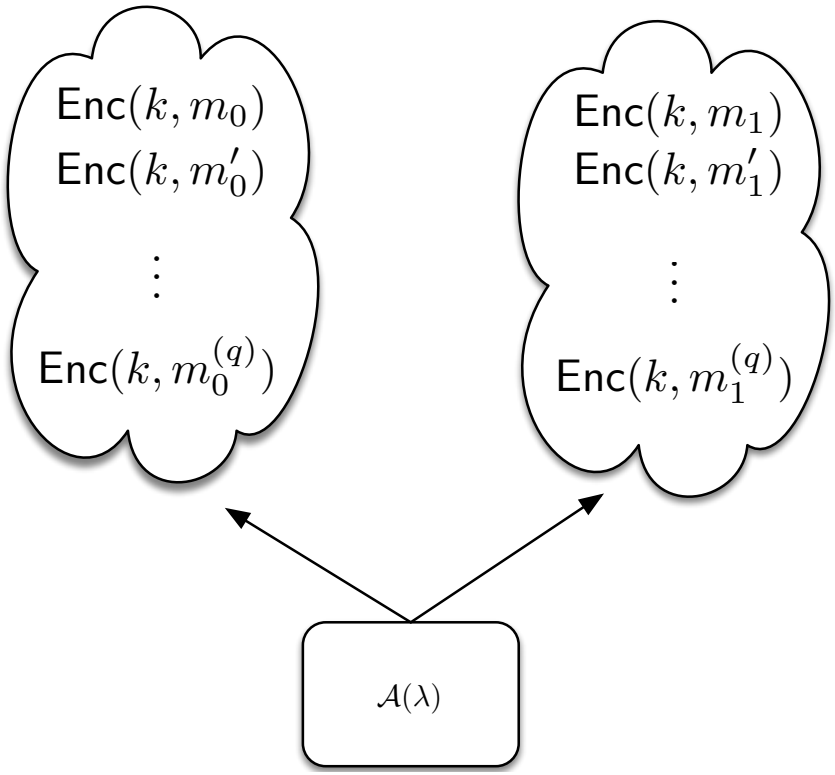
Attempt VI: A final tweak:

it's **infeasible** to find **any info.** about m
given **multiple** ciphertext with good **probability**.

Towards Formalizing Security



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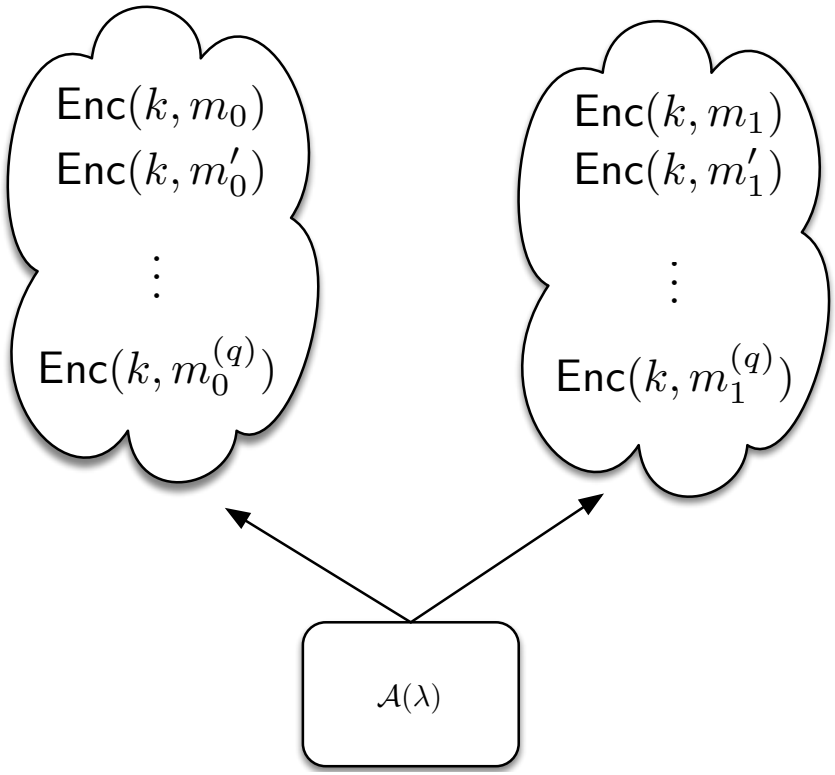

Ciphertexts look indistinguishable.

Experiments

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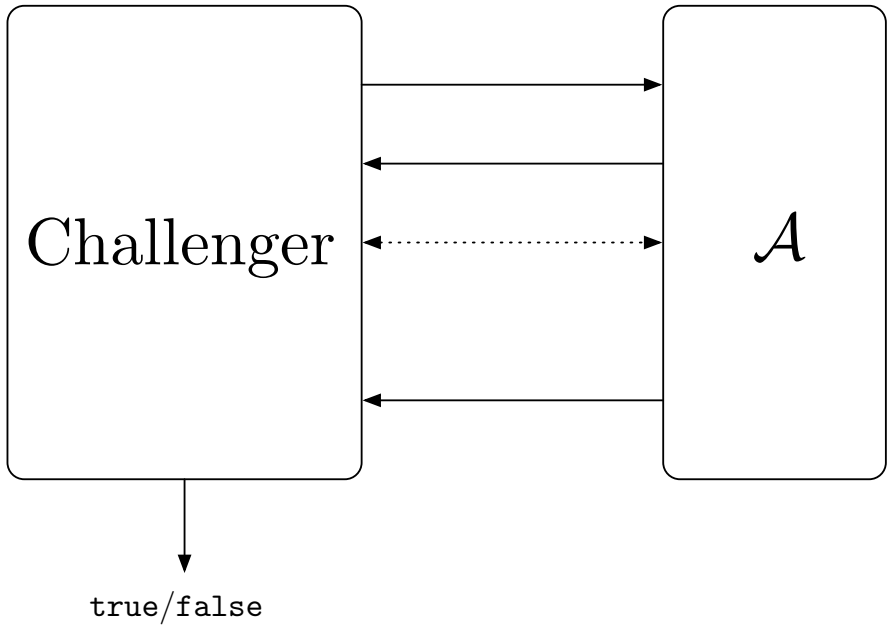

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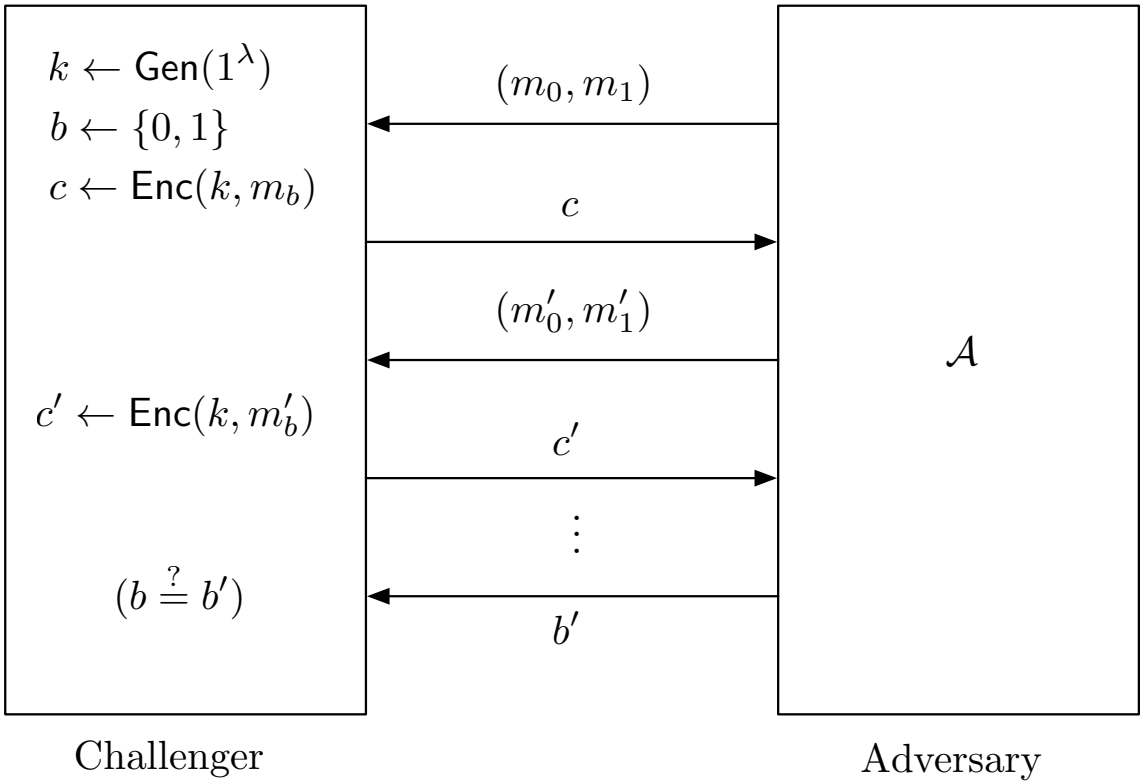

Formally

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Scheme SE is secure if:

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Note: Any encryption scheme meeting this notion must be **randomized**.

Two Quick Notes

Smallness:

A function $\mu : \mathbb{N} \longrightarrow \mathbb{R}$ is negligible/small if it goes to zero faster that $1/\lambda^c$ for any c:

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Efficiency:

A (randomized) algorithm $A(x)$ is efficient if there is a $c \in \mathbb{N}$ such that

$$(\forall x) \ : \ \textbf{Time}(A(x)) \leq |x|^c.$$

We call such A **probabilistic polynomial time**.

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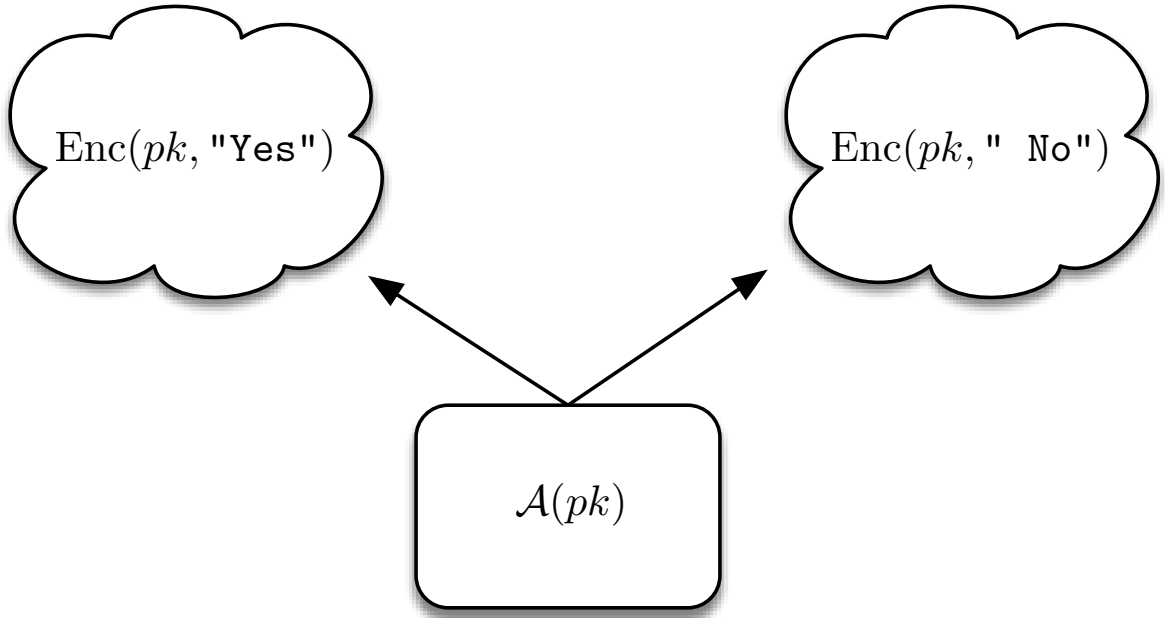
$$\mathrm{Negl} := \{ \text{ all negligible } \mu : \mathbb{N} \longrightarrow \mathbb{R} \ \}.$$

$$\mathrm{PPT} := \{ \text{ all efficient randomized Turing machines } \}.$$

Security of PKEs



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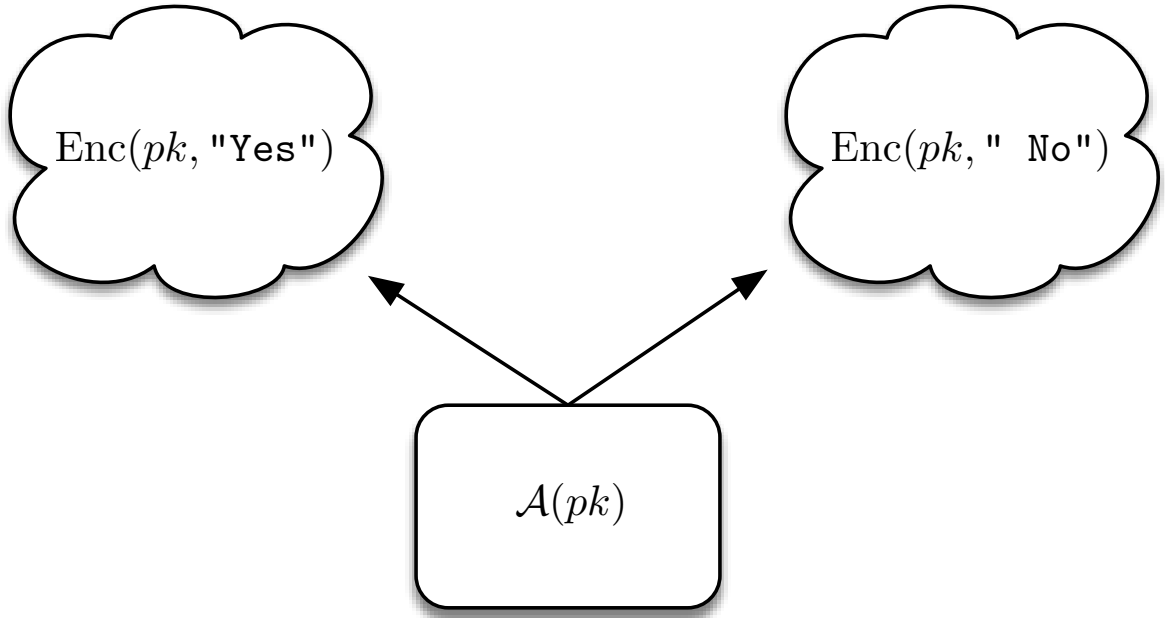
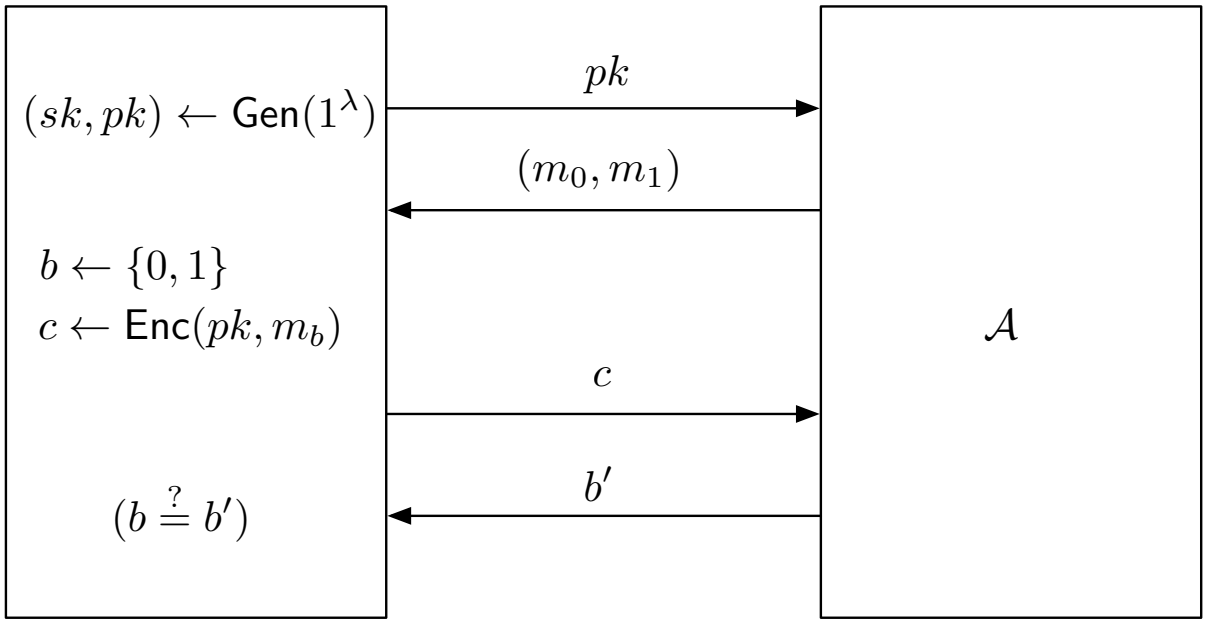


Ciphertexts look indistinguishable in the presence of pk.

Turns out: access to multiple ciphertexts doesn't affect the model. (Essentially because pk is public.)

Security of PKEs



Security of Public-Key Encryption

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Scheme PKE is IND-CPA secure if

$$(\forall \mathcal{A} \in \text{PPT}) \ : \ \mathbf{Adv}^{\text{ind-cpa}}_{\mathsf{PKE},\mathcal{A}}(\lambda) \in \text{Negl} \ .$$

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Contrapositive:

factoring is hard iff $\mathbf{Adv}^{\mathrm{factor}}_{\mathcal{Q}}(\lambda)$ is small for all efficient $\mathcal{Q}$.

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A cryptosystem is secure if

Any adversary of specified resources has small advantage in breaking the system.

Assumptions

(3) Assumptions.

Is Factoring Hard?

Recall:

- **P** := All problems that can be solved in polynomial time.
- **NP** := All problems whose solutions can be checked efficiently.

Whether or not

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$$\mathbf{P} = \mathbf{NP} \implies \text{factoring easy}$$

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remains **an/the** open problem in CS/Math. But,

$$\mathbf{P} = \mathbf{NP} \implies \text{factoring easy}$$
$$\text{i.e., factoring hard} \implies \mathbf{P} \neq \mathbf{NP}$$

Is Factoring Hard?

Recall:

- **P** := All problems that can be solved in polynomial time.
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Indeed:

$$\text{Secure SE, Secure PKE, } \ldots \implies \mathbf{P} \neq \mathbf{NP}.$$

So how can we deal with this?

Our Goal?

Our goal is:

To build highly secure and efficient cryptosystems.

As we just saw, we cannot prove hardness. Instead, we try to

Base the hardness of one cryptosystem/problem
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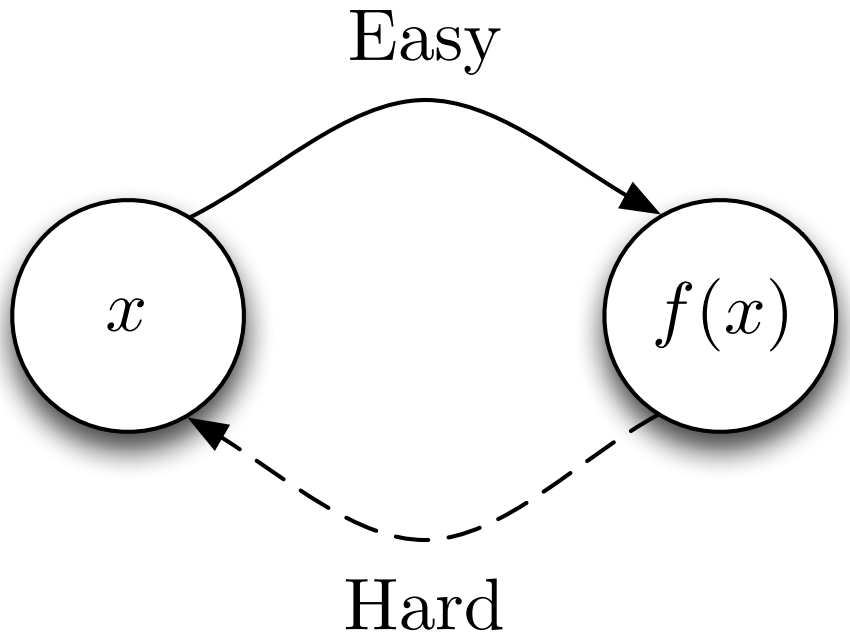
- hardness/security of P to be minimal/simple/well-studied.
- hardness/security of S to be maximal/realistic/novel.

A Minimal Hardness Notion: One-Way Functions



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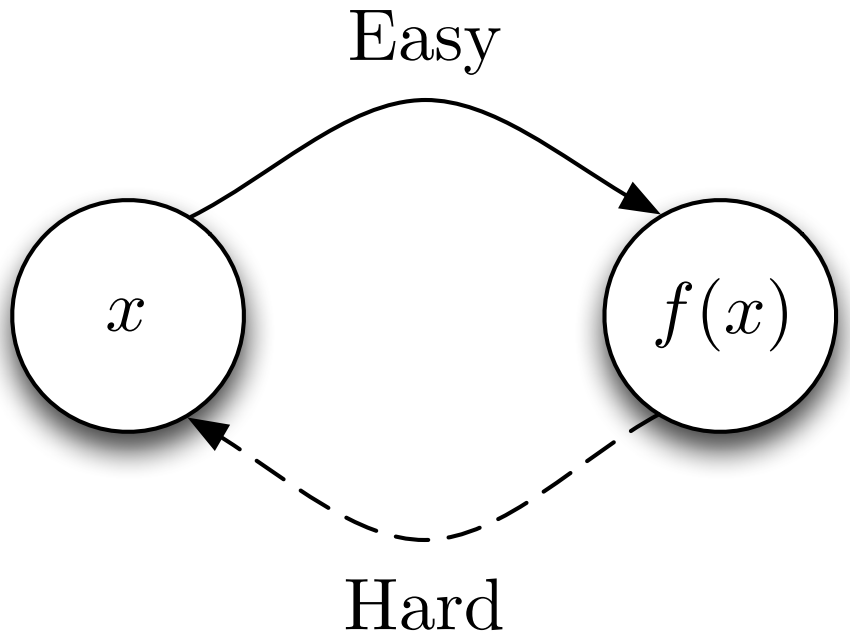

1. F(x): An efficient deterministic algorithm that on input a domain point x outputs an image point y.

One-Way Functions

Security: Hard to compute a preimage for $\mathsf{F}(x)$.

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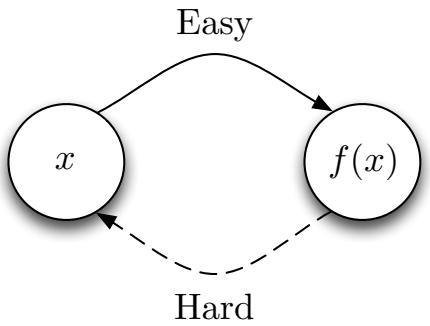

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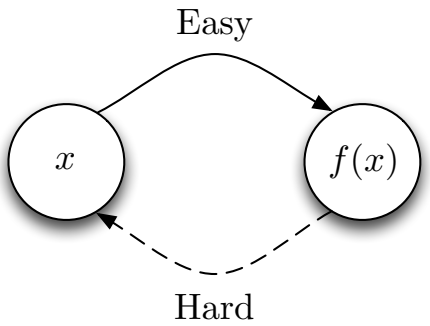

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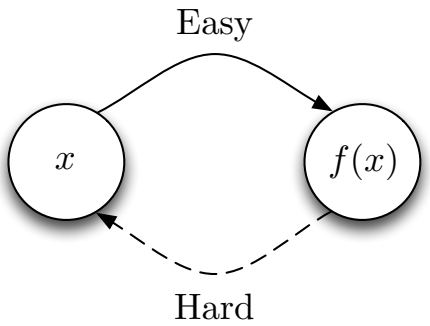

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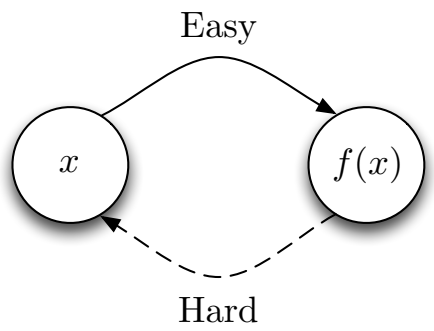

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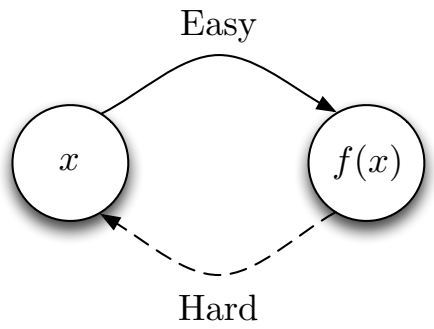

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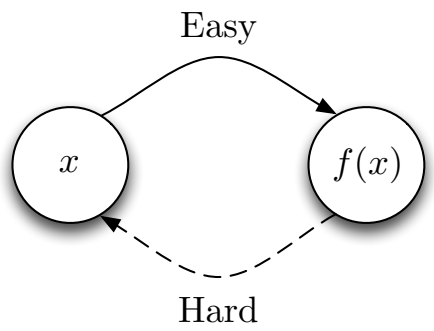

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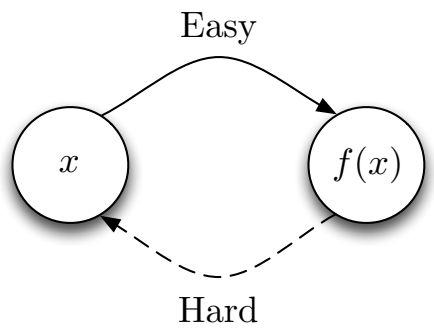

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Question 1: Is $f(x) := x \pmod 2$ one way?
Question 2: If factoring is hard, do OWFs exist?

Minimal Notion II: Collision-Resistance Hash Functions

Security: F for which it's hard to find $x_1 \neq x_2$ with $\mathsf{F}(x_1) = \mathsf{F}(x_2)$.
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Goal

Recall our goal was to:

Base the hardness of one cryptosystem/problem
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i.e., show:

$$\text{Problem } P \text{ hard} \implies \text{Cryosystem } S \text{ secure}$$

How can we prove this?

(4) Security Proofs via Reductions.

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Suppose a calculator can only square numbers. Can we use it to multiply?

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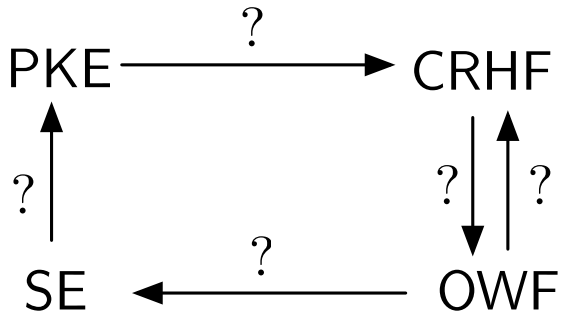
A **reduction** from M to S is an algorithm R^S that solves M using a subroutine for S.

Question: If I believe M is hard to compute and M reduces to S, does it follow that S is hard to compute?

Crypto

Can we reduce the security/correctness of one cryptosystem to the security/correctness of another?



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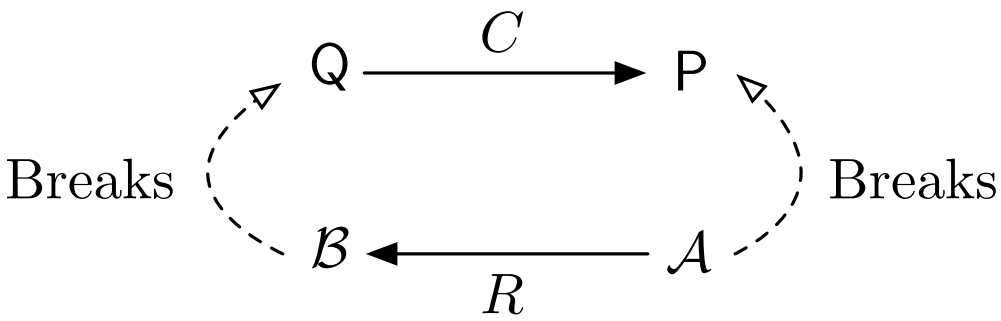

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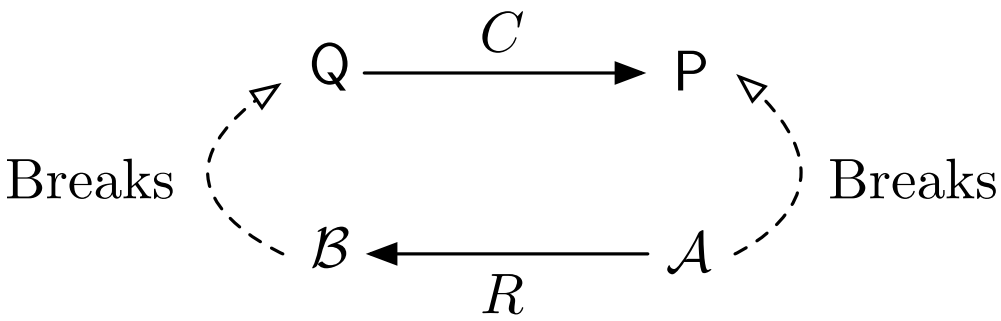

So if Q is secure, then so is P.

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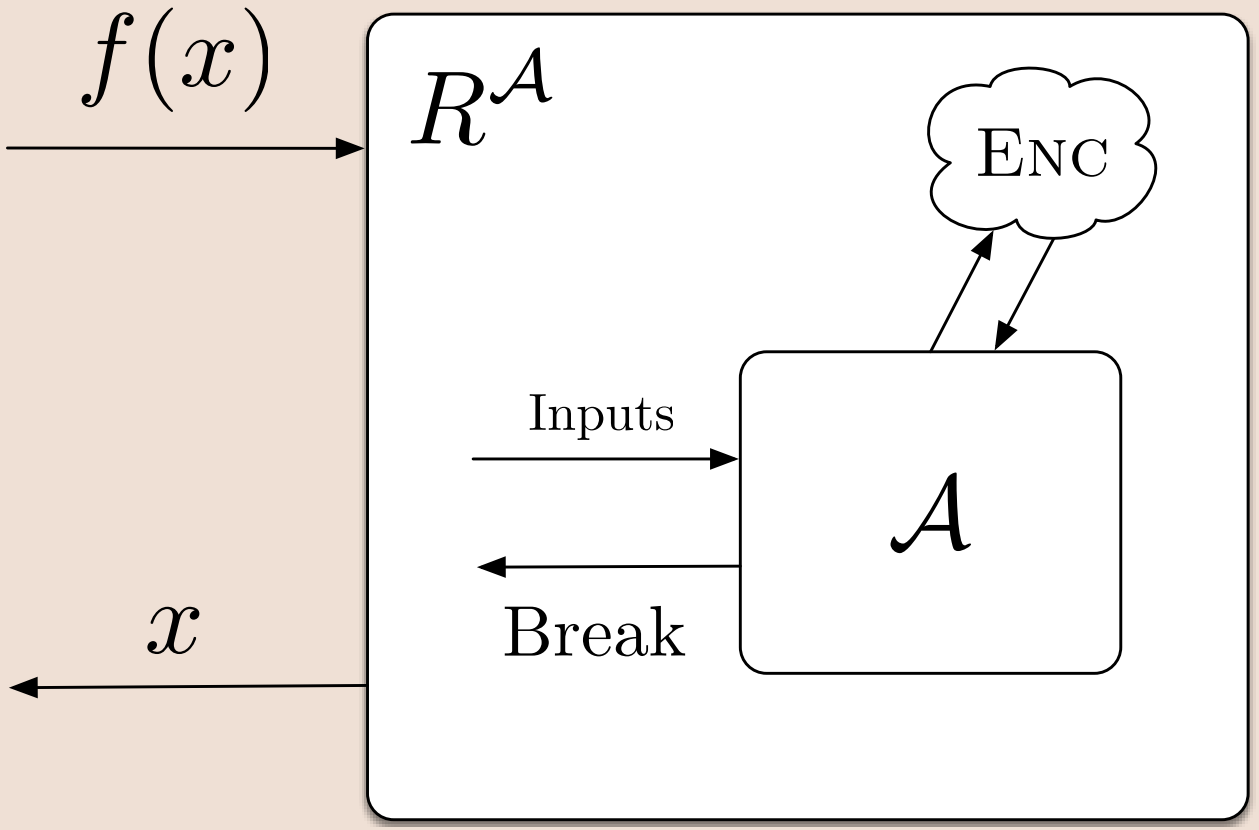
Guides setting key sizes.

Proof Template via Reductions

Proof.



Reductions

Example I: One-Way and Collision-Resistant Functions.

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Let's look at r-regular functions:

$$(\forall x) \ : \ |f^{-1}(f(x))| = r \qquad \text{for an } r > 1.$$

Note: r-regular functions are **compressing**.

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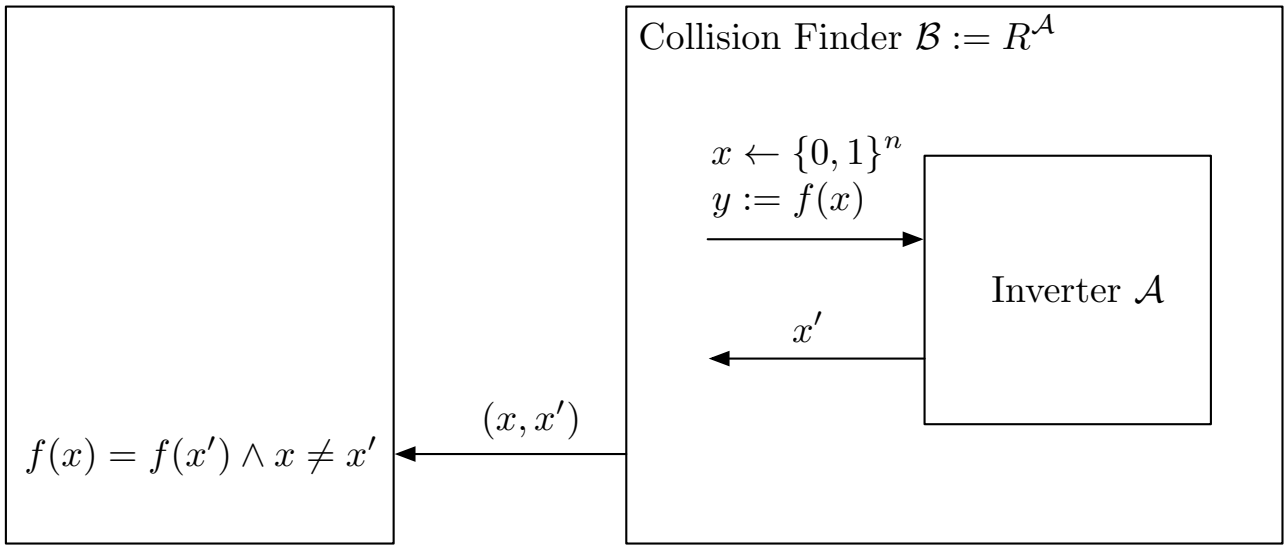

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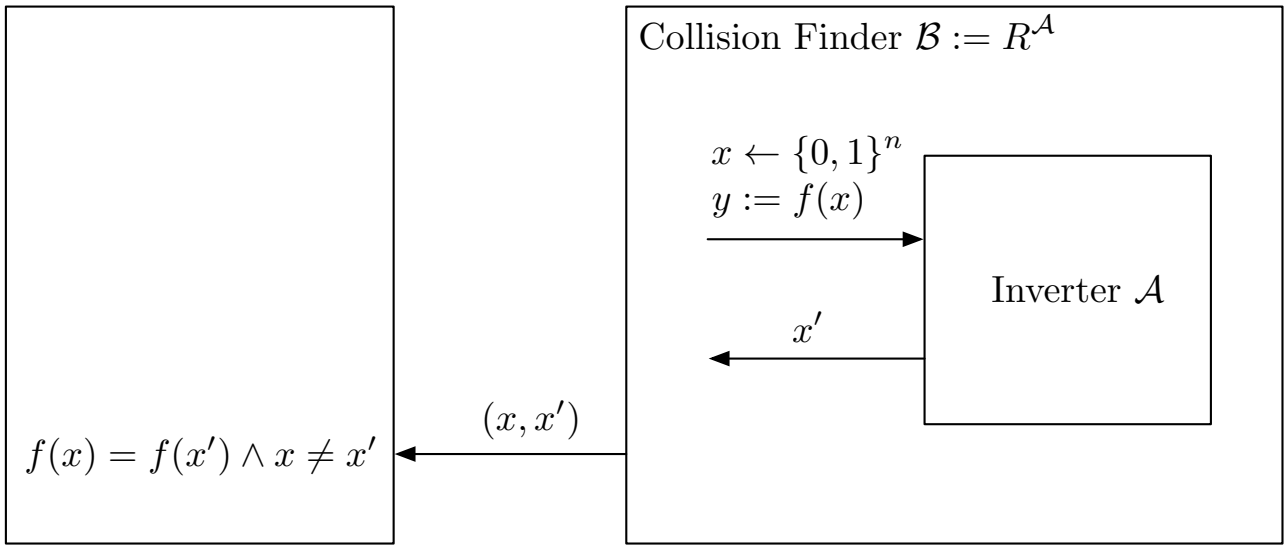

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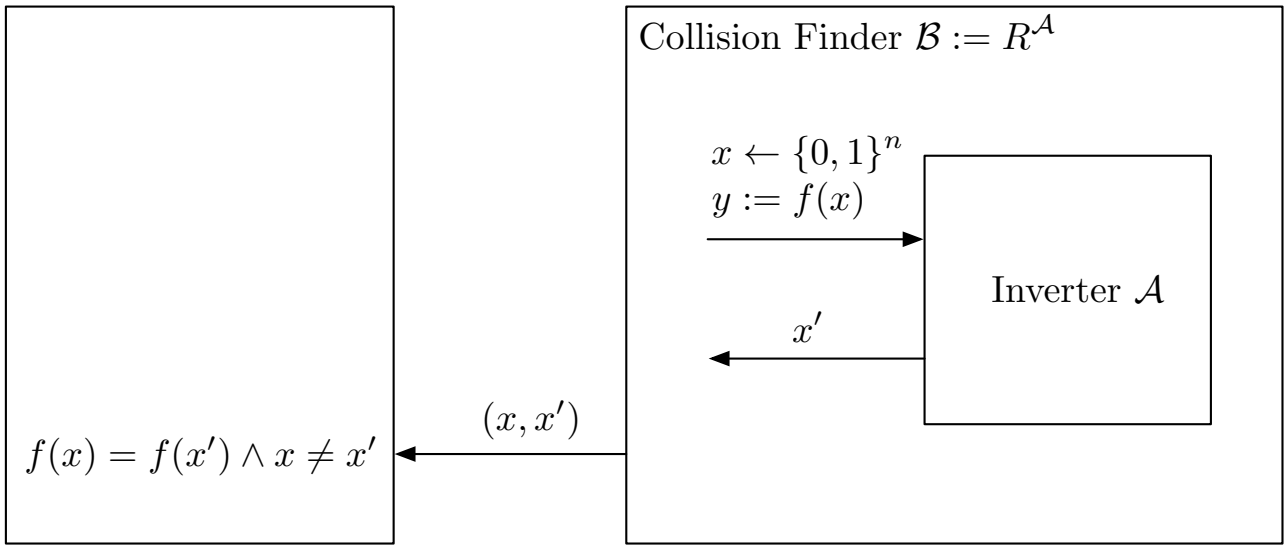

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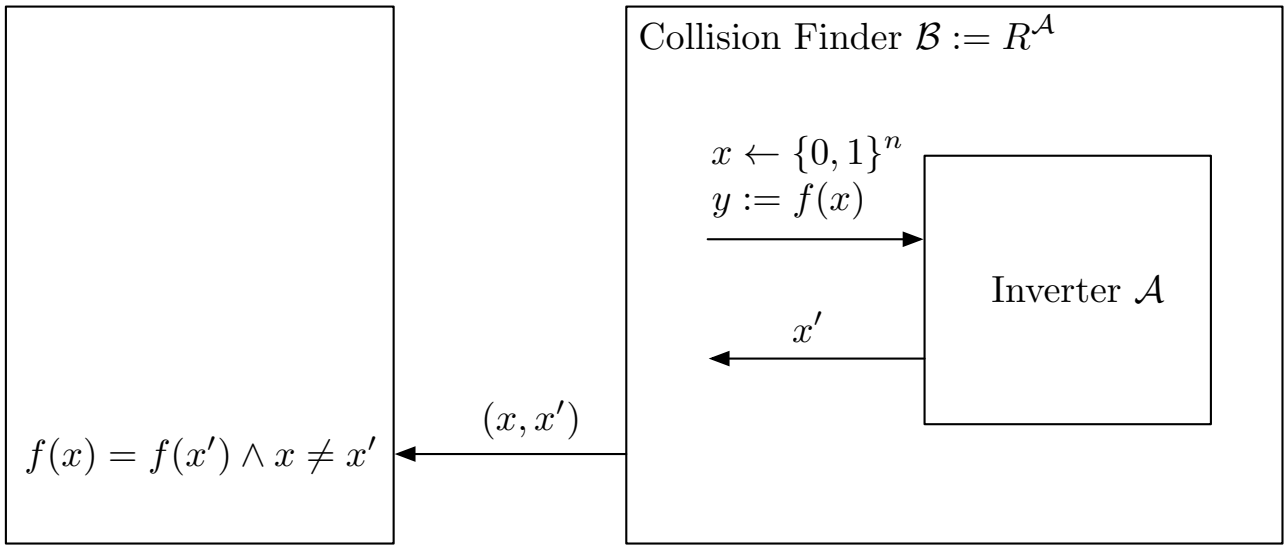

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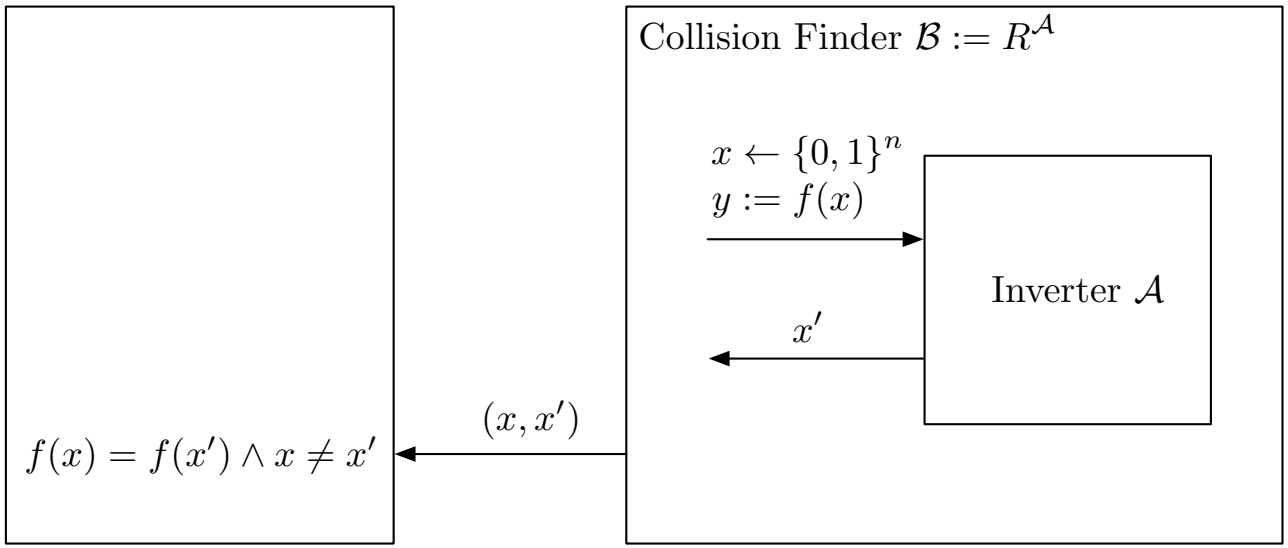

- Will $f(x') = f(x)$? With good prob. yes since $\mathcal{A}$ is a good inverter.
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Much more is known:

Allowing arbitrary C's does not help either [Sim98].

This is called an **impossibility** result.

Reducing Big CRHFs to Small CRHFs

Suppose we have build a CRHF with finite domain:

$$h : \{0,1\}^{2n} \longrightarrow \{0,1\}^{n}$$

How can we build a CRHF with arbitrary domain?

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How can we build a CRHF with arbitrary domain?

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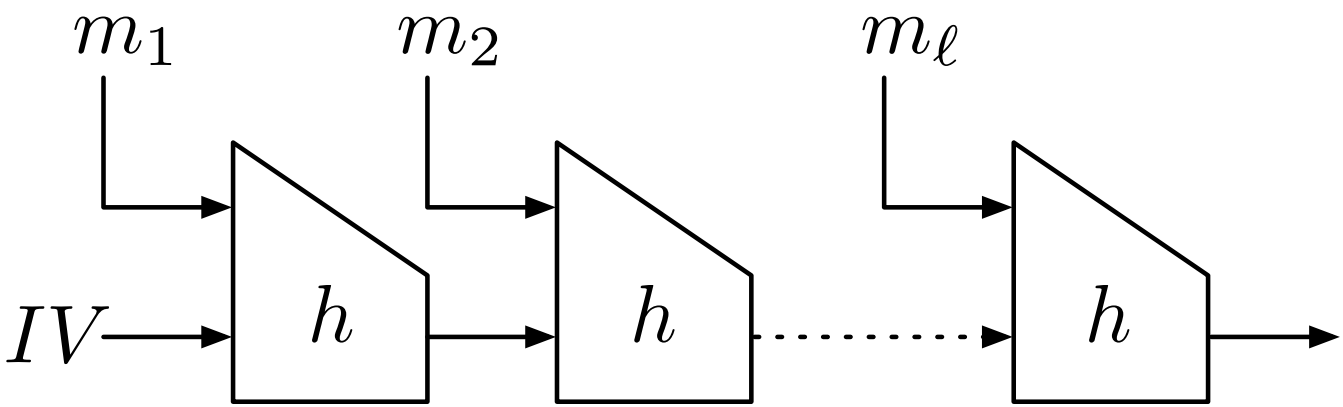
Merkle–Damgård chaining: Given $m = (m_1, m_2, \ldots, m_\ell)$ do:



Call this construction MD^h.

Reducing Big CRHFs to Small CRHFs

Theorem

If h is collision resistant then MD^h is also collision resistant.

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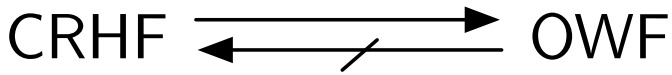
   - If $\underbrace{h(IV, m_1)}_{x_1} \neq \underbrace{h(IV, m_1')}_{x_1'}$: then $(x_1, m_2) \neq (x_1', m_2')$, a collision for h.

Summary



Example II: Encryption.

Groups

Recall cyclic groups $\mathbb{G} = \langle g \rangle$ of order p:

$$\mathbb{G} = \{g^a \ : \ a \in \mathbb{Z}_p\}, \quad g^a \cdot g^b = g^{a+b}.$$

One question for such groups is:

Given (g, g^x) can one efficiently compute x?

This is the **Discrete Logarithm** (DL) Problem.

DLP is a well-studied problem and is believed to be hard in certain groups.

Some Other Problems

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DDH Hard $\implies$ CDH Hard $\implies$ DL Hard

Can we use these problems to build a secure encryption scheme?

ElGamal Encryption

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$$\mathsf{Dec}(sk, (c_1, c_2)) = \frac{c_2}{c_1^{sk}} = \frac{m \cdot (g^x)^r}{(g^r)^x} = m$$

ElGamal Security

Is ElGamal secure?

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Theorem

If the DDH problem is hard, then ElGamal is IND-CPA secure.

As a result:

We have reduce the security of a PKE scheme with more complex syntax, correctness, and security to that of the DDH problem, which is simpler and better studied.

The DDH Assumption

Given (g, g^x, g^y, g^r) is $r = xy$?

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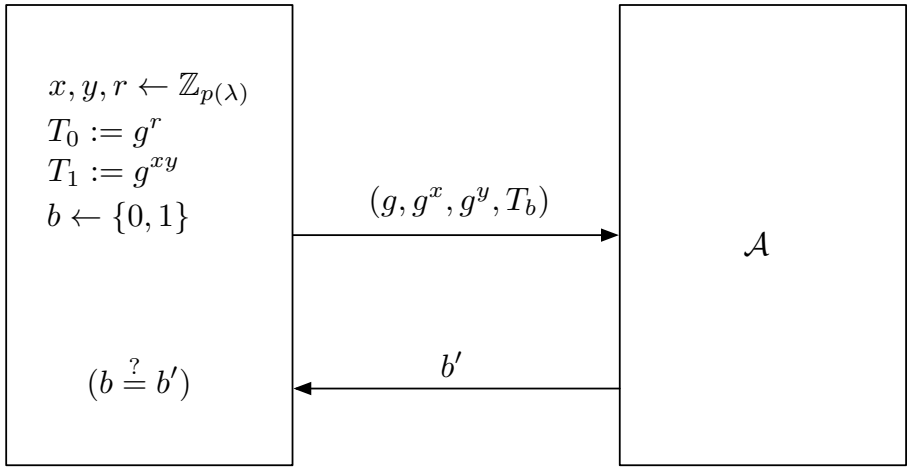

$$\mathbf{Adv}^{\mathrm{ddh}}_{G,\mathcal{A}}(\lambda) := \Pr[\mathrm{DDH}^{\mathcal{A}}_{G}(\lambda)] - 1/2$$

The IND-CPA Security of ElGamal

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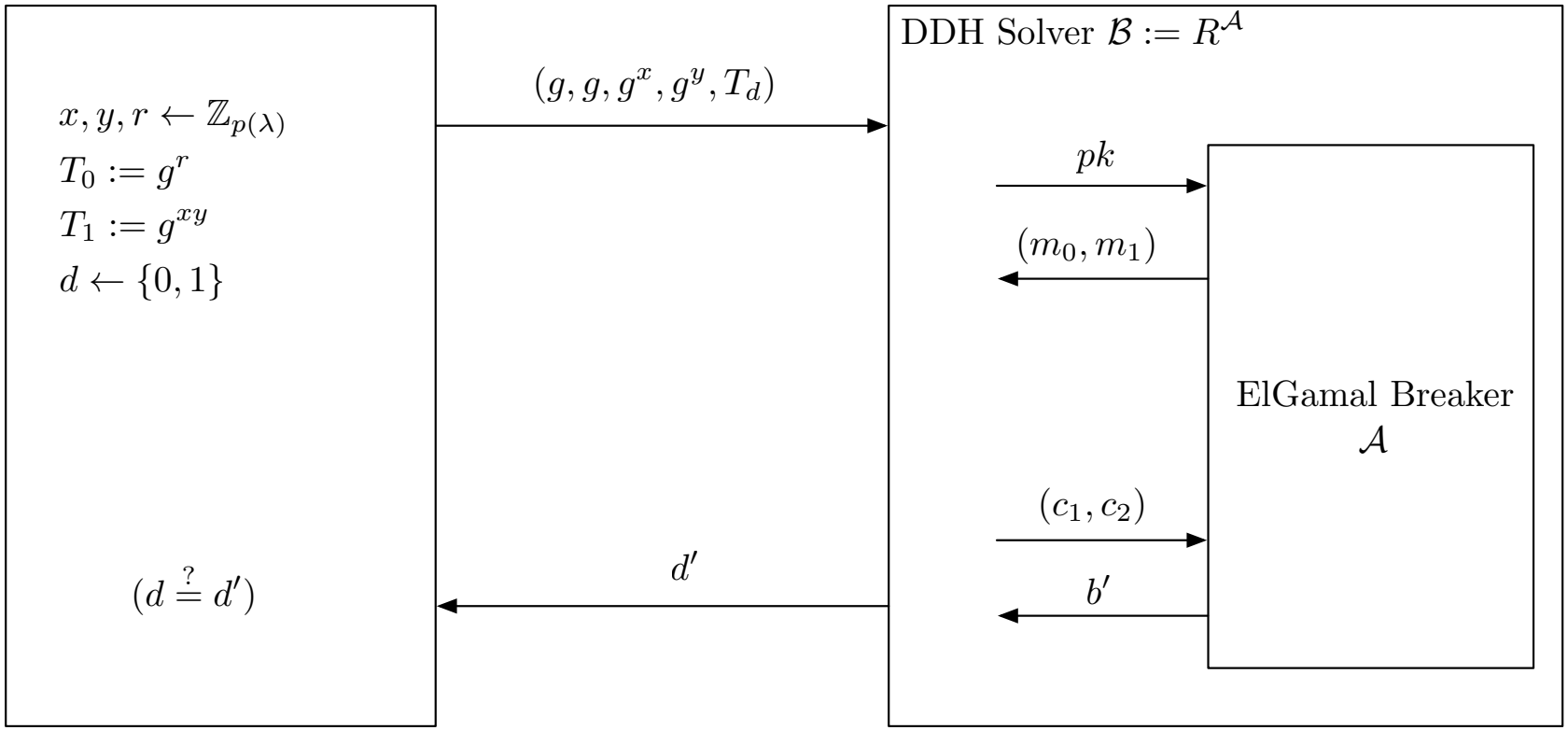

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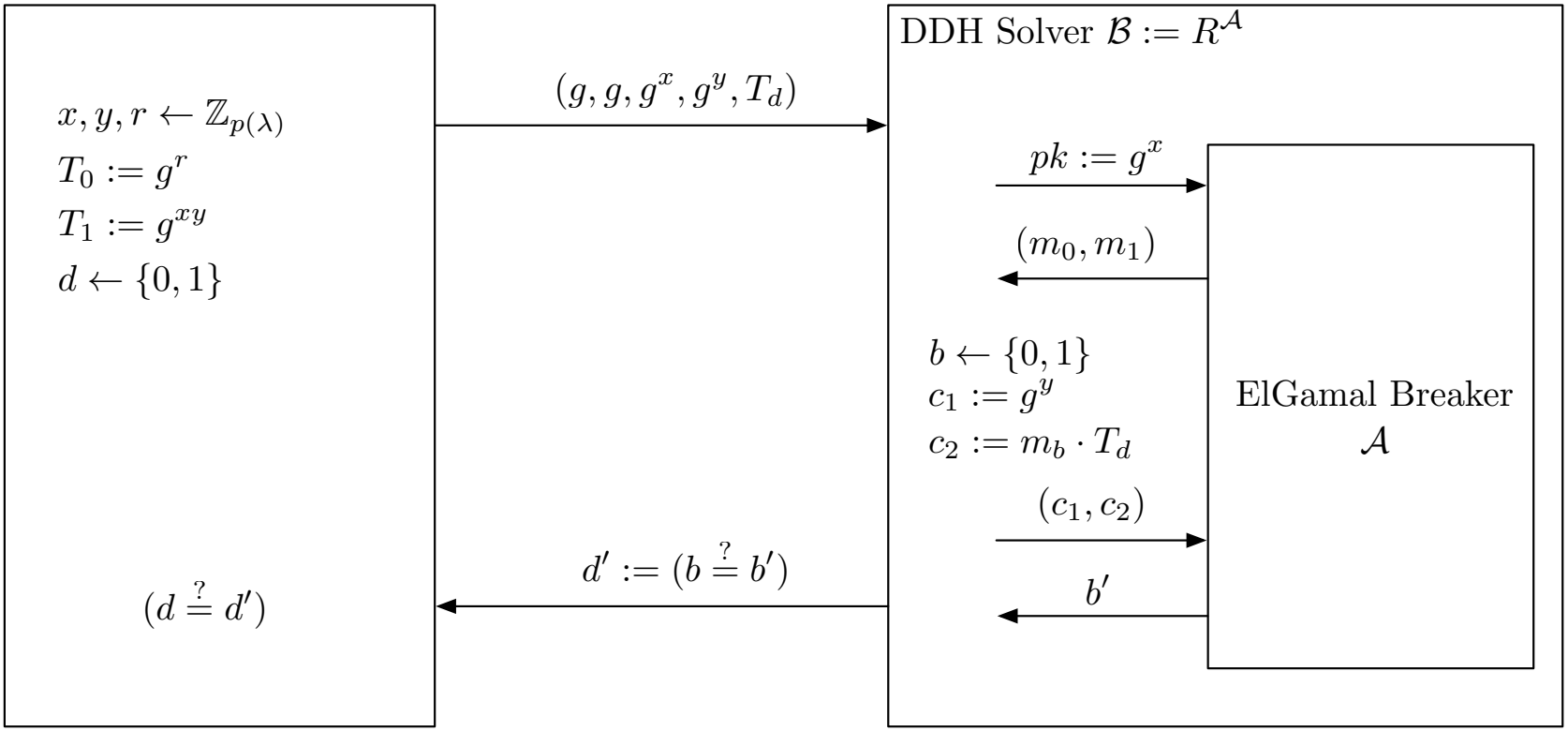
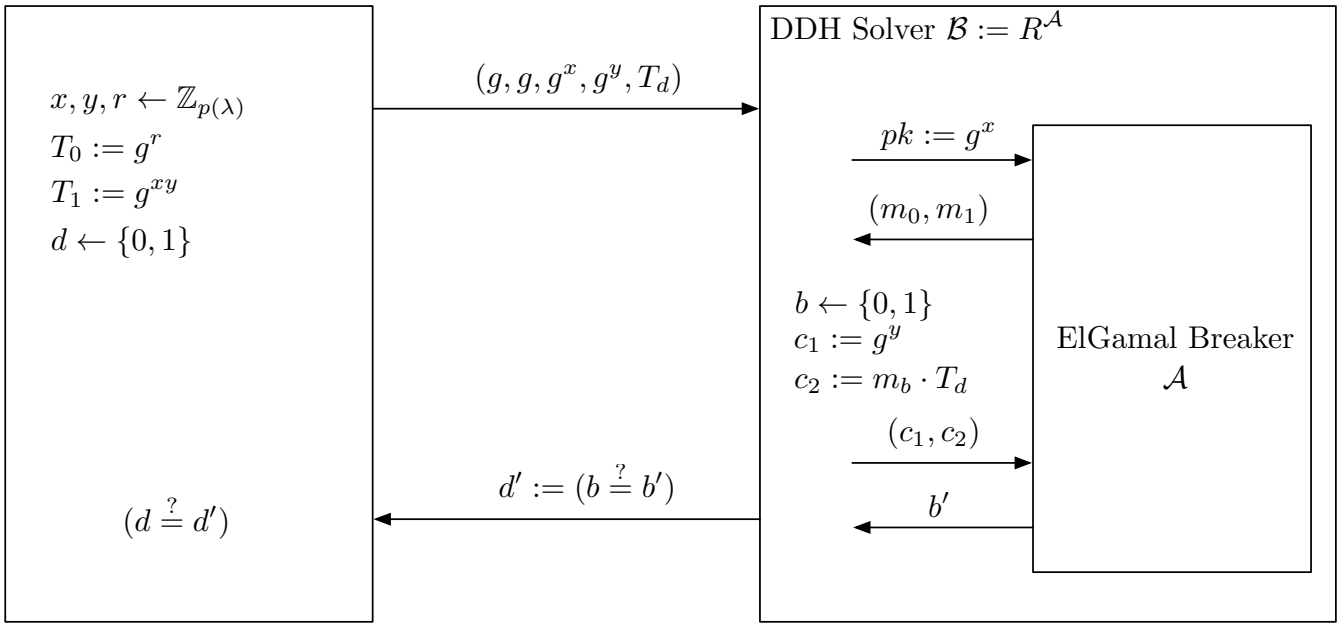


Proof


$x, y, r \leftarrow \mathbb{Z}_{p(\lambda)}$
$T_0 := g^r$
$T_1 := g^{xy}$
$d \leftarrow \{0,1\}$
$(d \stackrel{?}{=} d')$
(g, g, g^x, g^y, T_d)
$d' := (b \stackrel{?}{=} b')$
DDH Solver $\mathcal{B} := R^{\mathcal{A}}$
$pk := g^x$
(m_0, m_1)
$b \leftarrow \{0,1\}$
$c_1 := g^y$
$c_2 := m_b \cdot T_d$
(c_1, c_2)
b'
ElGamal Breaker
$\mathcal{A}$

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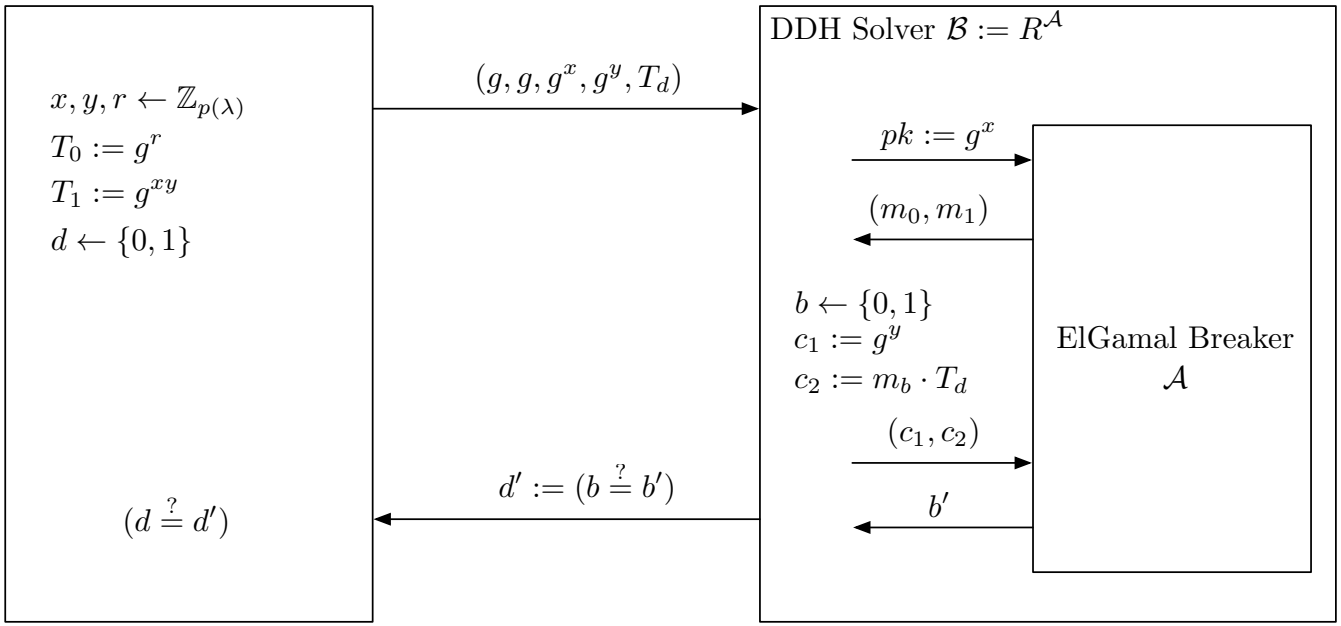

- $d = 1$: Then $T_d = g^{xy}$.

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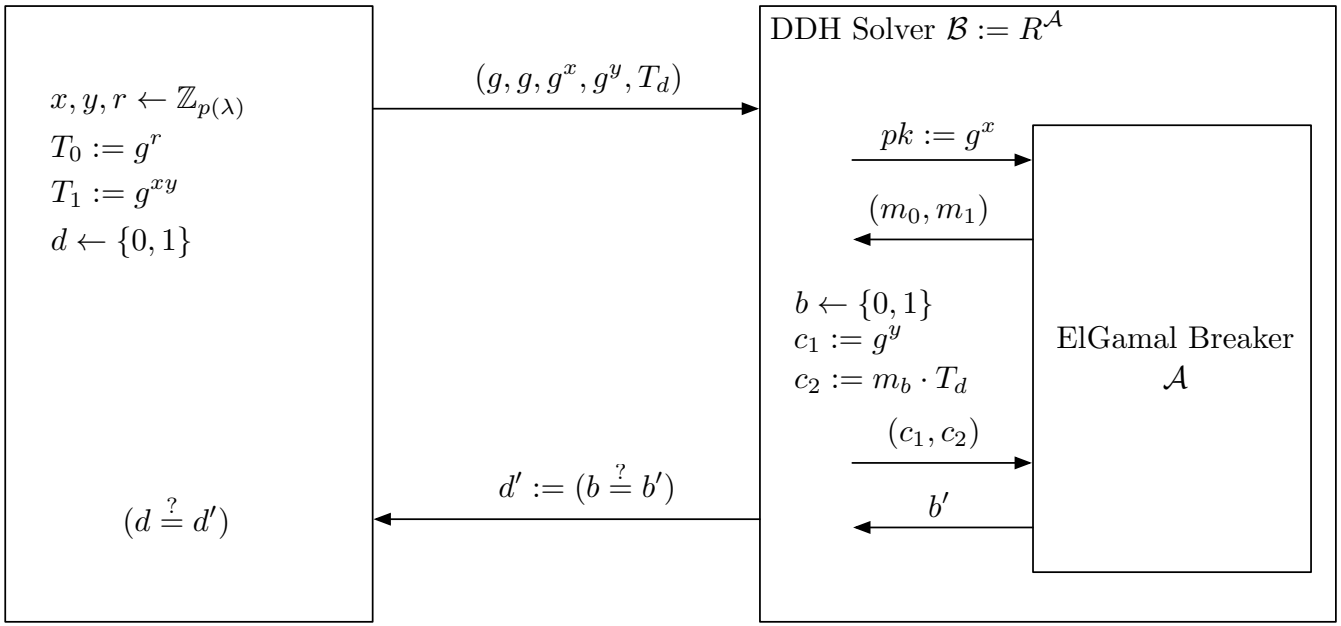

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$$\Pr[d = d' | d = 1] = \Pr[d' = 1] = \Pr[b = b'] = \mathbf{Adv}^{\text{ind-cpa}}_{\text{PKE},\mathcal{A}}(\lambda) + 1/2$$

- $d = 0$: Then $T_d = g^r$.

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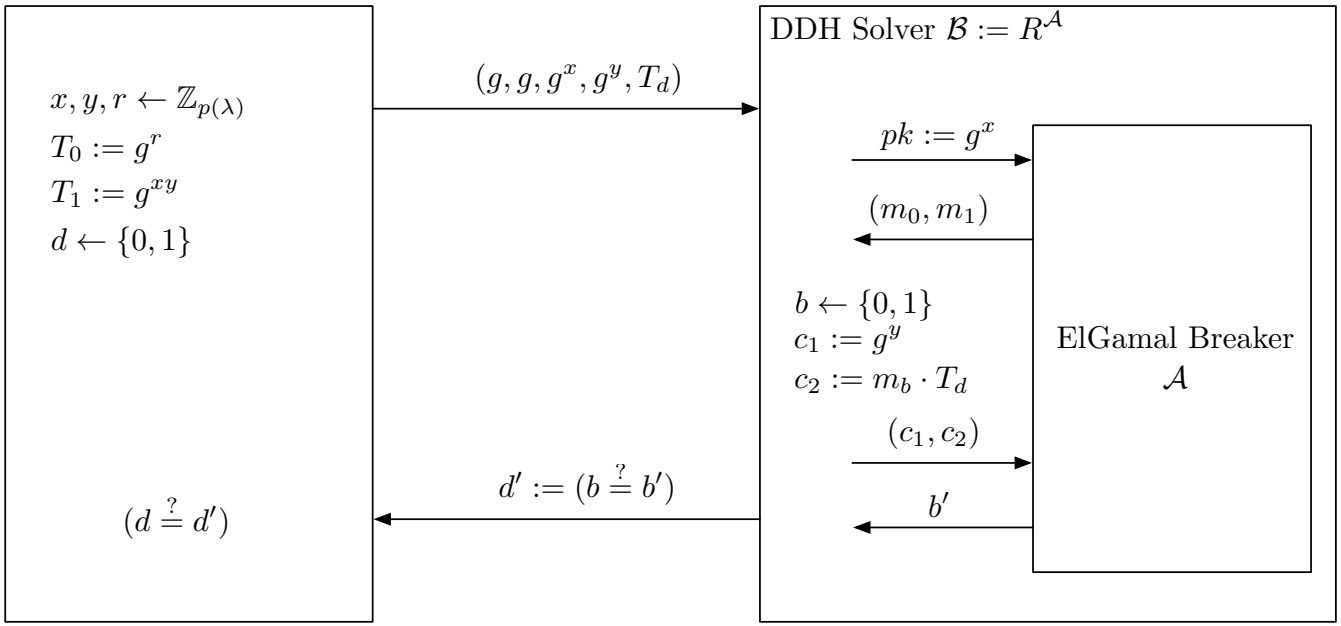

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- $d = 0$: Then $T_d = g^r$. Now $\mathcal{A}$ gets **no information** about b as a random group element g^r completely blinds m_b:

$$\Pr[d = d' | d = 0] = \Pr[b \neq b'] = 1/2 \ .$$

Proof

Continuing:

$$\begin{aligned}
\mathbf{Adv}^{\mathrm{ddh}}_{G,\mathcal{B}}(\lambda) &= \Pr[d = d'] - 1/2 \\
&= \Pr[d = d' | d = 1] \Pr[d = 1] + \Pr[d = d' | d = 0] \Pr[d = 0] - 1/2 \\
&= 1/2 \cdot (\Pr[d = d' | d = 1] + \Pr[d = d' | d = 0] - 1) \\
&= 1/2 \cdot (\mathbf{Adv}^{\text{ind-cpa}}_{\mathsf{PKE},\mathcal{A}}(\lambda) + 1/2 + 1/2 - 1) \\
&= 1/2 \cdot \mathbf{Adv}^{\text{ind-cpa}}_{\mathsf{PKE},\mathcal{A}}(\lambda)
\end{aligned}$$

What's going here?

- We are running the IND-CPA game with respect to T_d.
- If $d = 1$, and we have a good breaker, then $\Pr[b = b']$ will be noticeably above $1/2$.
- On the other hand, if $d = 0$, then $\Pr[b = b']$ will be exactly $1/2$ for **any** breaker.
- We use this difference in the two probabilities to decide d.

Proof Summary

DDH says:

$$(g, g^x, g^y, \underbrace{g^{xy}}_{d=1}) \quad \overset{c}{\approx} \quad (g, g^a, g^b, \underbrace{g^r}_{d=0})$$

The proof can be seen a transition

$$(\underbrace{g, g^x}_{pk}, \quad \underbrace{g^r, m_b \cdot g^{xy}}_{c}) \quad \rightarrow \quad (\underbrace{g, g^x}_{pk}, \quad \underbrace{g^r, m_b \cdot \boxed{g^r}}_{c})$$

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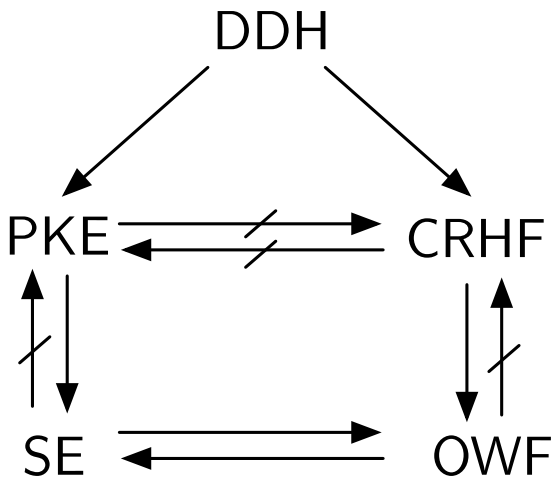
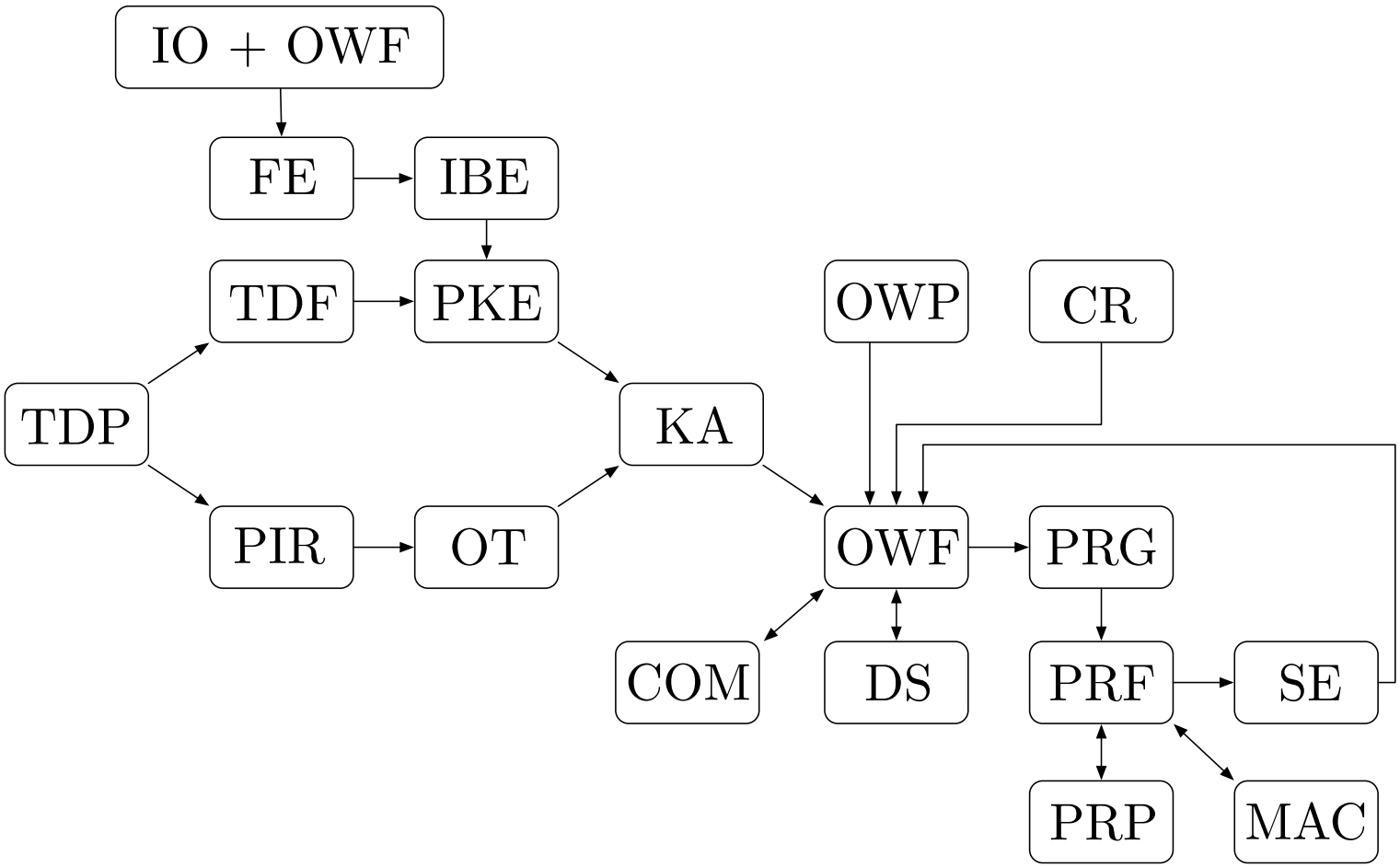
				Adv $= 0$

Landscape I



OWF is the weakest primitive, and hence the safest assumption.

Landscape II



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Caveats:

- Provable security never proves security in an absolute sense.
- Security is always relative to the assumptions and the model.
- Definitions, assumptions, constructions and proofs evolve.

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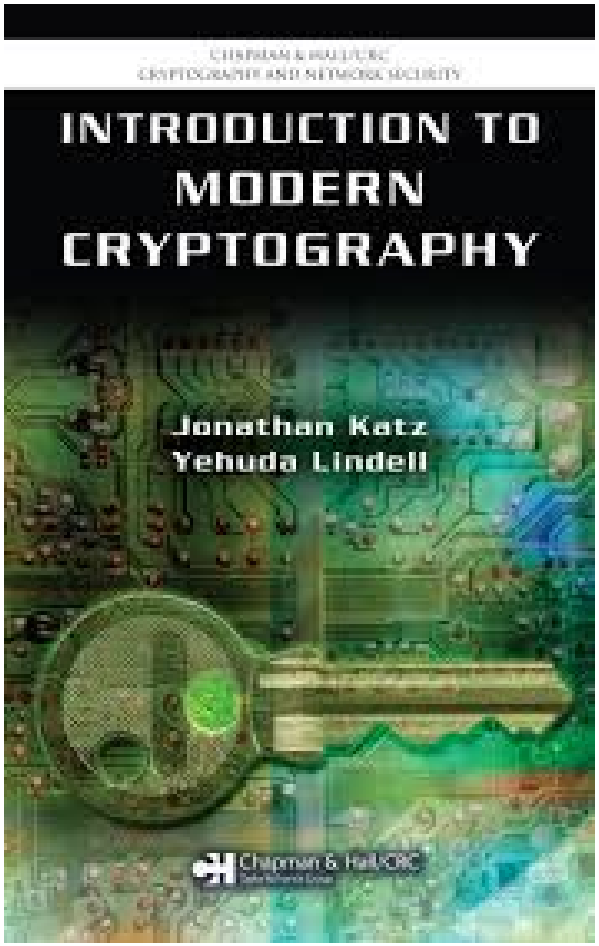
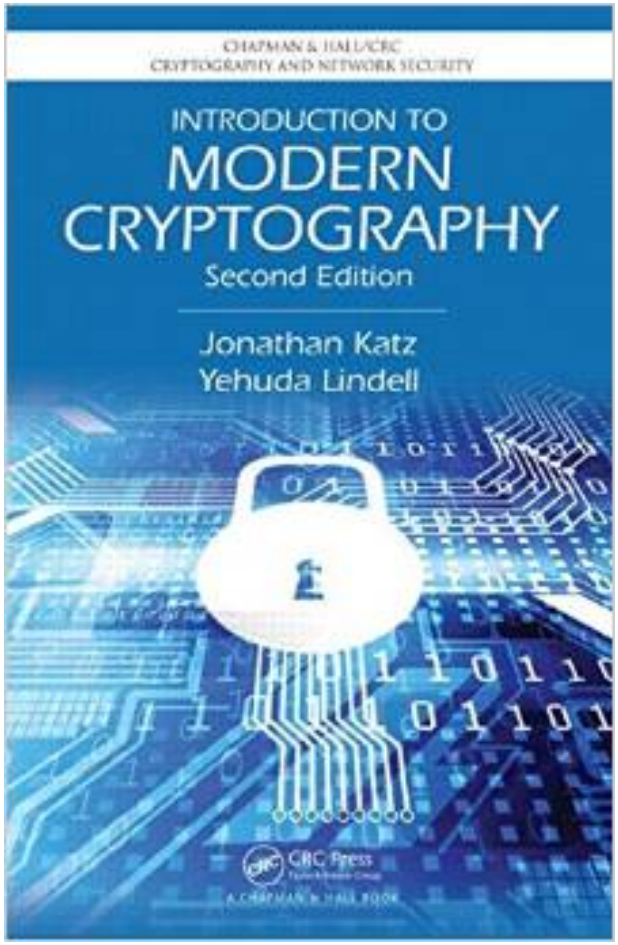
7. And of course: is the crypto task well-motivated?

Further Reading





Further Viewing

Further Reading

1998 Practice-Oriented Provable-Security. (Bellare)

2004 On the Role of Definitions in and Beyond Cryptography. (Rogaway)

2004 Another look at "Provable Security". (Kobliz & Menezes)

2006 On Post-Modern Cryptography. (Goldreich)

2009 Practice-Oriented Provable Security and the Social Construction of Cryptography. (Rogaway)

2015 The Moral Character of Cryptographic Work. (Rogaway)

Further Reading

1998 Practice-Oriented Provable-Security. (Bellare)

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2004 Another look at "Provable Security". (Kobliz & Menezes)

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Thanks.